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# A characterization of maximal non-$k$-factor-critical graphs

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**Abstract**

A graph $G$ of order $p$ is $k$-factor-critical,where $p$ and $k$ are positive integers with the same parity, if the deletion of any set of $k$ vertices results in a graph with a perfect matching. $G$ is called maximal non-$k$-factor-critical if $G$ is not $k$-factor-critical but $G+e$ is $k$-factor-critical for every missing edge $e \notin E(G)$. A connected graph $G$ with a perfect matching on $2n$ vertices is $k$-extendable, for $1 \leqslant k \leqslant n-1$, if for every matching $M$ of size $k$ in $G$ there is a perfect matching in $G$ containing all edges of $M$. $G$ is called maximal non-$k$-extendable if $G$ is not $k$-extendable but $G+e$ is $k$-extendable for every missing edge $e \notin E(G)$ . A connected bipartite graph $G$ with a bipartitioning set $(X, Y)$ such that $|X|=|Y|=n$ is maximal non-$k$-extendable bipartite if $G$ is not $k$-extendable but $G+xy$ is $k$-extendable for any edge $xy \notin E(G)$ with $x \in X$ and $y \in Y$. A complete characterization of maximal non-$k$-factor-critical graphs, maximal non-$k$-extendable graphs and maximal non-$k$-extendable bipartite graphs is given.


*Keywords:* Matching; $k$-factor-critical graphs; $k$-extendable graphs

## 1. Introduction

All graphs considered in this paper are finite, connected, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [2]. Thus $G$ is a graph with vertex set $V(G)$, edge set $E(G)$ and minimum degree $\delta(G)$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph induced by $V'$. Similarly, $G[E']$ denotes the subgraph induced by the edge set $E'$of $G$. $N_G(u)$ denotes the neighbour set of $u$ in $G$ and $\overline{N}_G(u)$ the non-neighbours of $u$. Note that $\overline{N}_G(u) = V(G) \backslash (N_G(u) \cup \{u\})$. The join $G \vee H$ of disjoint graphs $G$ and $H$ is the graph obtained from $G \cup H$ by joining each vertex of $G$ to each vertex of $H$.

A *matching* $M$ in $G$ is a subset of $E(G)$ in which no two edges have a vertex in common. A vertex $v$ is saturated by $M$ if some edge of $M$ is incident to $v$; otherwise $v$ is said to be unsaturated. A matching $G$ is *perfect* if it saturates every vertex of $G$. For simplicity, we let $V(M)$ denote the vertex set of the subgraph $G[M]$ induced by $M$. A graph $G$ of order $p$ is *$k$-factor-critical*, where $p$ and $k$ are positive integers with the same parity, if the deletion of any set of $k$

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vertices results in a graph with a perfect matching. $G$ is called *maximal non-k-factor-critical* if $G$ is not $k$-factor-critical but $G+e$ is $k$-factor-critical for every missing edge $e \notin E(G)$. The concept of $k$-factor-critical is a generalization of the concepts of factor critical and bicritical. $k$-factor critical graphs have been studied by many authors including Favaron [3,4], Favaron and Shi [6,7] and Favaron et al. [5].

A closely related concept to $k$-factor-critical is that of *k-extendable*. For $1 \leqslant k \leqslant n-1$, a connected graph $G$ of order $2n$ with a perfect matching is $k$-extendable if for every matching $M$ of size $k$ in $G$ there is a perfect matching in $G$ containing all edges of $M$. For convenience, a graph $G$ with a perfect matching is said to be 0-extendable. $G$ is called *maximal non-k-extendable* if $G$ is not $k$-extendable but $G+e$ is $k$-extendable for every missing edge $e \notin E(G)$. A connected bipartite graph $G$ with a bipartitioning set $(X, Y)$ such that $|X|=|Y|=n$ is *maximal non-k-extendable bipartite* if $G$ is not $k$-extendable but $G+xy$ is $k$-extendable for any edge $xy \notin E(G)$ with $x \in X$ and $y \in Y$. Extendable graphs have been studied by many authors including Plummer [9], Ananchuen and Caccetta [1], Kawarabashi et al. [8], Ryjáček [12] and Yu [14]. Excellent surveys are the papers of Plummer [10,11] . In this paper, we introduce the concepts of maximal non-$k$-factor-critical, maximal non $k$-extendable and maximal non $k$-extendable bipartite graphs.

A $2k$-factor-critical graph is obviously $k$-extendable but the converse need not be true, since a complete bipartite graph $K_{n,n}$ is $k$-extendable for $0 \leqslant k \leqslant n-1$ but is not $2k$-factor-critical. Further, the graph $G$ formed by joining two $K_{2k}$'s with a perfect matching is $k$-extendable non-bipartite but is not $2k$-factor-critical. On the other hand, the graphs $G_1$ and $G_2$, shown in Fig. 1, are both maximal non-2-extendable graphs and maximal non-4-factor-critical graphs whilst the graphs $G_3$ and $G_4$, shown in Fig. 2, are both maximal non-2-extendable bipartite graphs, since the edge $u_1v_1$ together with the edge $u_2v_2$ cannot extend to a perfect matching in each $G_i$ for $1 \leqslant i \leqslant 4$. Note that these graphs are 1-extendable. This is no coincidence as it is true in general, a fact we will establish later. However, the definitions of maximal non-$k$-factor-critical, maximal-non-$k$-extendable and maximal-non-$k$-extendable bipartite graphs give no suggestion of this property.

Further, the above examples suggest that there may be a relationship between maximal non-$k$-factor-critical graphs and maximal non-$k$-extendable graphs. In this paper, we establish the strong connection between these two classes of graphs. More precisely, we establish that for a connected graph $G$ on $2n$ vertices with a perfect matching, $G$ is maximal non-$k$-extendable if and only if $G$ is maximal non-$2k$-factor-critical for $1 \leqslant k \leqslant n-1$. We also provide a characterization of

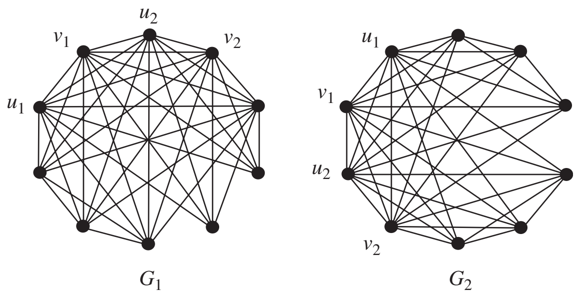


Fig. 1.

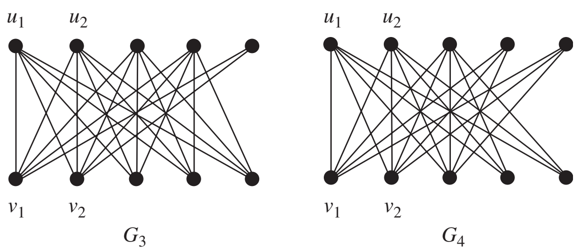


Fig. 2.

maximal non-$k$-factor-critical graphs, maximal non-$k$-extendable graphs and maximal non-$k$-extendable bipartite graphs.

## 2. Maximal non-$k$-factor-critical graphs

In this section, we establish a characterization of maximal non-$k$-factor-critical graphs. We begin with the following lemma.

**Lemma 2.1.** *For positive integers $p$ and $k$ having the same parity, and non-negative integers $s, t_1, t_2, \ldots, t_{s+2}$ with $0 \leqslant s \leqslant \frac{1}{2}(p-k)-1$ and $\sum_{i=1}^{s+2} t_i = \frac{1}{2}(p-k)-s-1$, the graph*

$$G = K_{k+s} \vee \bigcup_{i=1}^{s+2} K_{2t_i+1}$$

*is maximal non-$k$-factor-critical of order $p$.*

**Proof.** Let $H = K_{k+s}$ and $G_i = K_{2t_i+1}$ for $1 \leqslant i \leqslant s+2$. Then $G = H \vee \bigcup_{i=1}^{s+2} G_i$. Let $T$ be a subset of $V(H)$ with $|T| = k$. Clearly, $G - T = K_s \vee \bigcup_{i=1}^{s+2} G_i$ has no perfect matching. Thus $G$ is not $k$-factor-critical.

We next show that $G$ is maximal. Let $u$ and $v$ be non-adjacent vertices in $G$ and let us consider $G' = G + uv$. Clearly, $u$ and $v$ are vertices of $G_i$ and $G_j$ for some $i \neq j$, respectively. Let $T'$ be a subset of $V(G')$ with $|T'| = k$ and let $r = |V(H) \cap T'|$.

*Case* 1: $r = k$.

Clearly, $G' - T'$ has a perfect matching containing the edge $uv$.

*Case* 2: $r = k - 1$.

Then exactly $s+1$ of the subgraphs $G_i - T'$ have odd order. Since $H - T'$ has order $s+1$, $G' - T'$ has a perfect matching.

*Case* 3: $r \leqslant k - 2$.

Suppose exactly $t$ of the subgraphs $G_i - T'$ have odd order. Then $t$ and the order of $H - T'$ have the same parity. Also the order of $H - T'$ is $k + s - r \geqslant s + 2 \geqslant t$. Hence $G' - T'$ has a perfect matching.

Therefore, $G' = G + uv$ is $k$-factor-critical and hence $G$ is maximal non-$k$-factor-critical. □

Before we establish a characterization of maximal non-$k$-factor-critical graphs we recall Tutte's Theorem which we make use of in our proof. As usual we let $o(H)$ denote the number of odd components in $H$.

**Theorem 2.2.** *Tutte's Theorem* (*see Bondy and Murty* [2, p. 76]). *A graph $G$ has a perfect matching if and only if $o(G - S) \leqslant |S|$ for all $S \subset V(G)$.*

Now we are ready for our main theorem in this section.

**Theorem 2.3.** *Let $G$ be a connected graph on $p$ vertices and $k$ a positive integer having the same parity with $p$. $G$ is maximal non-$k$-factor-critical if and only if*

$$G \cong K_{k+s} \vee \bigcup_{i=1}^{s+2} K_{2t_i+1}$$

*where $s$ and $t_i$ are non-negative integers with $0 \leqslant s \leqslant \frac{1}{2}(p-k)-1$ and $\sum_{i=1}^{s+2} t_i = \frac{1}{2}(p-k)-s-1$.*

**Proof.** The sufficiency follows from Lemma 2.1. Now we prove the necessity. Since $G$ is maximal non-$k$-factor-critical, there is a subset $T$ of $V(G)$ of size $k$ such that $G' = G - T$ has no perfect matching. Then, by Tutte's Theorem, there is a subset $S'$ of $V(G')$ such that $o(G' - S') > |S'|$. Put $s = |S'|$. Because $G'$ is of even order, it follows that $s$ and $o(G' - S')$ must have the same parity. Thus $o(G' - S') \geqslant s + 2$.

Let $C_1, C_2, \ldots, C_r$ be odd components of $G' - S'$. We first show that $r = s + 2$. Suppose to the contrary that $r \geqslant s + 3$. Then $r \geqslant s + 4$. Let $c_i \in V(C_i)$ for $i = 1, 2$ and let us consider $G + c_1c_2$. Clearly, $(G + c_1c_2) - (T \cup S')$ contains at least $s + 2$ odd components. Thus $G + c_1c_2$ is not $k$-factor-critical. This contradicts the fact that $G$ is maximal non-$k$-factor-critical. Hence, $r = s + 2$ as required.

We next show that $G' - S'$ has no even components. Suppose to the contrary that $G' - S'$ contains $D$ as an even component. Let $d \in D$ and $c_1 \in V(C_1)$. Now consider $G + dc_1$. Clearly, $(G + dc_1) - (T \cup S')$ contains exactly $s + 2$ odd components since the components $D$ and $C_1$ together with the edge $dc_1$ forms an odd component of $G + dc_1$. Thus $G + dc_1$ is not $k$-factor-critical, a contradiction. This proves that $G' - S'$ has no even components.

Now we claim that $G[T \cup S']$ is complete. Suppose it is not the case. Then there exist vertices $x$ and $y$ in $T \cup S'$ such that $xy \notin E(G)$. Now consider $G + xy$. Since $(G + xy) - (T \cup S')$ contains exactly $s + 2$ odd components, $G + xy$ is not $k$-factor-critical. This contradiction proves that $G[T \cup S']$ is complete. By a similar argument, it is easy to establish that each $C_i$ is complete for $1 \leqslant i \leqslant s + 2$. Further, for $1 \leqslant i \leqslant s + 2$, each vertex of $C_i$ is adjacent to every vertex of $T \cup S'$.

Now, for $1 \leqslant i \leqslant s+2$, let $|V(C_i)| = 2t_i + 1$ for some non-negative integer $t_i$. Then $p = |V(G)| = k + s + \sum_{i=1}^{s+2} |V(C_i)| = k + 2s + 2 + 2\sum_{i=1}^{s+2} t_i \geqslant k + 2s + 2$. Hence, $\sum_{i=1}^{s+2} t_i = \frac{1}{2}(p - k) - s - 1$ and $0 \leqslant s \leqslant \frac{1}{2}(p - k) - 1$ as required. This completes the proof of our theorem. □

As a corollary we have:

**Corollary 2.4.** *If $G$ is a maximal non-$k$-factor-critical graph on $p$ vertices where $k$ is a positive integer greater than* 1 *having the same parity with $p$, then $G$ is $(k - 2)$-factor-critical.*

## 3. Maximal non-$k$-extendable graphs

In this section, we characterize maximal non-$k$-extendable graphs and show that they are closely related to maximal non-$k$-factor-critical graphs.

**Theorem 3.1.** *Let $G$ be a connected graph with a perfect matching on $2n$ vertices. For $1 \leqslant k \leqslant n - 1$, $G$ is maximal non-$k$-extendable if and only if*

$$G \cong K_{2k+s} \vee \bigcup_{i=1}^{s+2} K_{2t_i+1}$$

*where $s$ and $t_i$ are non-negative integers with $0 \leqslant s \leqslant n - k - 1$ and $\sum_{i=1}^{s+2} t_i = n - k - s - 1$.*

**Proof.** The sufficiency follows from Lemma 2.1 and the definitions of factor-critical graphs and $k$-extendable graphs. For the necessity, the proof is almost identical with the proof in Theorem 2.3 so we omit it. □

As a corollary we have:

**Corollary 3.2.** *Let $G$ be a maximal non-$k$-extendable graph on $2n$ vertices for $1 \leqslant k \leqslant n - 1$. Then $G$ is $(k - 1)$-extendable.*

**Corollary 3.3.** *Let $G$ be a maximal non-$k$-extendable graph on $2n$ vertices for $1 \leqslant k \leqslant n - 1$. If $E' \subseteq E(K_{2n}) \backslash E(G)$ with $|E'| \geqslant 1$, then $G + E'$ is $k$-extendable.*

**Proof.** The result follows by applying a similar argument as in the proof of Lemma 2.1 to the graph $G + E'$. □

**Remark 3.1.** (1) A connected graph with a perfect matching which is not $k$-extendable need not be $(k-1)$-extendable. For example, a cycle on $2n \geqslant 8$ vertices is not 3-extendable and it is not 2-extendable. In the case of a maximal non-$k$-extendable graph $G$, $G$ is not $k$-extendable but it is $(k - 1)$-extendable. This is not immediately obvious but it can be proved in a straight forward way from the definitions.

(2) In [14] Yu proved that if $G$ is a $k$-extendable graph on $2n$ vertices with $1 \leqslant k \leqslant n-1$, then $G+e$ is $(k-1)$-extendable for any edge $e \notin E(G)$. Hence, adding a new edge into a $k$-extendable graph $G$ might destroy the $k$-extendability property of $G$. This is not so for a maximal non-$k$-extendable graph, however, no matter how many edges are added. The resulting graph is still $k$-extendable providing that the number of edges is at least 1.

By Theorems 2.3 and 3.1, we have the theorem.

**Theorem 3.4.** *Let $G$ be a connected graph on $2n$ vertices with a perfect matching. For $1 \leqslant k \leqslant n-1$, $G$ is maximal non-$k$-extendable if and only if $G$ is maximal non-$2k$-factor-critical.*

**Remark 3.2.** As we mentioned in the Introduction that $k$-extendable graphs need not be $2k$-factor-critical but for a maximal non $k$-extendable graph $G$, $G+e$ is both $k$-extendable and $2k$-factor-critical for any edge $e \notin E(G)$.

**Remark 3.3.** A variation of $k$-extendability is that of induced matching extendability or IM-extendability for short which was introduced by Yuan [15]. A matching $M$ of $G$ is induced if $E([V(M)]) = M$. A graph $G$ is IM-extendable if every induced matching of $G$ is included in a perfect matching of $G$. Notice that an IM-extendable graph is 1-extendable. Further, a $k$-extendable graph with no induced matching of size greater than $k$ is IM-extendable. Wang and Yuan [13] introduced a concept of maximal IM-unextendable graphs. A graph $G$ is called maximal IM-unextendable if it is not IM-extendable but $G+xy$ is IM-extendable for every two non-adjacent vertices $x$ and $y$ of $G$. They established that the only maximal IM-unextendable graph is $M_k \vee (K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{s+2}}))$ where $M_k$ is an induced matching of size $k \geqslant 1$, $s$ is a non-negative integer and each $n_i$ is odd. Observe that the class of maximal IM-unextendable graphs coincides with the class of maximal non-$k$-extendable graphs only for $k=1$.

## 4. Maximal non-$k$-extendable bipartite graphs

In this section, we extend our idea on maximal non-$k$-extendable graphs to the case of bipartite graphs as follows. Let $G$ be a connected bipartite graph on $2n$ vertices with a bipartitioning set $(X, Y)$ such that $|X| = |Y| = n$. For non-negative integers $k$ and $n$ with $0 \leqslant k \leqslant n-1$, $G$ is *maximal non-$k$-extendable bipartite* if $G$ is not $k$-extendable but $G+e$ is $k$-extendable for any edge $e = xy \notin E(G)$ where $x \in X$, $y \in Y$. Thus we are interested in adding a missing edge $e \notin E(G)$, which has one of its end vertices in $X$ and the other in $Y$. We also establish a characterization of maximal non-$k$-extendable bipartite graphs. We first recall Hall's Theorem.

**Theorem 4.1.** *Hall's Theorem (see Bondy and Murty [2, p. 72]). Let $G$ be a bipartite graph with bipartitioning $(X, Y)$. Then $G$ contains a matching that saturates every vertex in $X$ if and only if $|N(S)| \geqslant |S|$ for all $S \subseteq X$.*

**Lemma 4.2.** *For any non-negative integers $n$, $k$ and $s$ with $1 \leqslant s \leqslant n-1$ and $2 \leqslant k+s \leqslant n$, let $(X, Y)$ be a bipartitioning set of $K_{n,n}$ and let $S \subseteq X$, $T \subseteq Y$ with $|S| = s$ and $|T| = n-k-s+1$. Then*

$$G = K_{n,n} - \{xy | x \in S, y \in T\}$$

*is a maximal non-$k$-extendable bipartite graph on $2n$ vertices.*

**Proof.** The result is obvious for $k=0$. We have to consider only $k \geqslant 1$. Let $M$ be a matching of size $k$ in $G$ consisting of edges $e_i = u_i v_i \in M$, $u_i \in X \backslash S$, $v_i \in Y \backslash T$ for $1 \leqslant i \leqslant k$. Then $S \subseteq X \backslash V(M)$ with $|N_{G-V(M)}(S)| = s-1 < s = |S|$. Thus, $G - V(M)$ has no perfect matching by Hall's Theorem. Hence, $G$ is not $k$-extendable.

Now we establish that $G$ is maximal. Let $e = xy \notin E(G)$ where $x \in X$ and $y \in Y$. Clearly, $x \in S$ and $y \in T$.

Consider $G' = G + xy$. Let $M'$ be a matching of size $k$ in $G'$ and

$$k_1 = |(X \backslash S) \cap V(M')|, \quad k_2 = |S \cap V(M')|,$$

$$k_3 = |(Y \backslash T) \cap V(M')| \quad \text{and} \quad k_4 = |T \cap V(M')|.$$

Then $k_1 + k_2 = k = k_3 + k_4$, $|X\backslash(S \cup V(M')| = n - k_1 - s$ and $|Y\backslash(T \cup V(M')| = k + s - 1 - k_3$. We distinguish two cases according to $k_1$.

*Case* 1: $k_1 = k$.

Clearly, $k_2 = 0$ and $|S\backslash V(M')| = s$.

*Subcase* 1.1: $k_4 = 0$. Then $k_3 = k$ and $xy \in E(G' - V(M'))$. There is a matching $M_1'$ of $G' - V(M')$ of size $s - 1$ joining vertices of $S\backslash\{x\}$ to vertices of $Y\backslash(T \cup V(M'))$ and a matching $M_2'$ of $G' - V(M')$ of size $n - k - s$ joining vertices of $T\backslash\{y\}$ to vertices of $X\backslash(S \cup V(M'))$. Now $G' - V(M')$ contains $M_1' \cup M_2' \cup \{xy\}$ as a perfect matching as required.

*Subcase* 1.2: $k_4 \geqslant 1$. Then $k_3 \leqslant k - 1$. Thus $s \leqslant k + s - 1 - k_3$. Now let $M_1''$ be a matching of $G' - V(M')$ of size $s$ joining vertices of $S$ to vertices of $Y\backslash(T \cup V(M'))$. Further, let $M_2''$ be a matching of $G' - V(M')$ of size $n-k-s+1-k_4$ joining vertices of $T\backslash V(M')$ to vertices of $X\backslash(S \cup V(M'))$. Now $G - V(M' \cup M_1'' \cup M_2'') \cong K_{m,m}$, where $m = k_4 - 1$, contains a perfect matching $M_3''$. Hence, $M_1'' \cup M_2'' \cup M_3''$ forms a perfect matching of $G' - V(M')$.

*Case* 2: $k_1 \leqslant k - 1$.

Then $k_2 \geqslant 1$. Further, $n - k - s + 1 \leqslant n - k_1 - s$ and $s - k_2 \leqslant s - 1 \leqslant k - k_3 + s - 1$. Now let $M_1'''$ be a matching of $G' - V(M')$ of size $s - k_2$ joining vertices of $S\backslash V(M')$ to vertices of $Y\backslash(T \cup V(M'))$. Further, let $M_2'''$ be a matching of $G' - V(M')$ of size $n - k - s + 1 - k_4$ joining vertices of $T\backslash V(M')$ to vertices of $X\backslash(S \cup V(M'))$. Now $G - V(M' \cup M_1''' \cup M_2''') \cong K_{m,m}$, where $m = k_2 + k_4 - 1$, contains a perfect matching $M_3'''$. Hence, $M_1''' \cup M_2''' \cup M_3'''$ is a perfect matching of $G' - V(M')$. Therefore, $G' = G + xy$ is $k$-extendable as required. This completes the proof of our lemma.

Now we establish the main result of this section.

**Theorem 4.3.** *Let $G$ be a connected bipartite graph on $2n$ vertices with a bipartitioning set $(X, Y)$ such that $|X| = |Y|$. For $0 \leqslant k \leqslant n - 1$, $G$ is maximal non-$k$-extendable bipartite if and only if there are subsets $S \subseteq X$, $T \subseteq Y$ with $|S| = s$ and $|T| = n - k - s + 1$ such that*

$$G \cong K_{n,n} - \{xy | x \in S, y \in T\}$$

*for an integer $s$ with $1 \leqslant s \leqslant n - 1$ and $2 \leqslant k + s \leqslant n$.*

**Proof.** The sufficiency follows from Lemma 4.2. So we need only prove the necessity. Since $G$ is maximal non-$k$-extendable bipartite, there is a matching $M$ of size $k$ in $G$ such that $G - V(M)$ has no perfect matching. Let $(X', Y')$ be a bipartitioning set of $G' = G - V(M)$. Clearly, $X' = X\backslash V(M)$ and $Y' = Y\backslash V(M)$. Further, $|X'| = n - k = |Y'|$. Since $G'$ has no perfect matching, by Hall's Theorem, there is a subset $S \subseteq X'$ such that $s = |S| \geqslant |N_{G'}(S)| + 1 \geqslant 1$. Clearly, $s \leqslant n - k$. We next show that $s = |N_{G'}(S)| + 1$. Suppose to the contrary that $s \geqslant |N_{G'}(S)| + 2$. Then $|Y'\backslash N_{G'}(S)| = n - k - |N_{G'}(S)| \geqslant n - k - s + 2 \geqslant 2$. Let $x \in S$ and $y \in Y'\backslash N_{G'}(S)$. Clearly, $xy \notin E(G)$. But$(G + xy) - V(M) = G' + xy$ contains $S$ as a subset of $X'$ with $s = |S| > (s - 2) + 1 \geqslant |N_{G'}(S)| + 1 = |N_{G'+xy}(S)|$. Thus, $(G + xy) - V(M)$ has no perfect matching. Hence, $G + xy$ is not $k$-extendable. This contradicts the fact that $G$ is maximal non-$k$-extendable bipartite. Therefore, $s = |N_{G'}(S)| + 1$.

We next show that each vertex of $S$ is adjacent to every vertex of $(V(M) \cap Y) \cup N_{G'}(S)$. Suppose this is not the case. Then there are vertices $a \in S$ and $b \in (V(M) \cap Y) \cup N_{G'}(S)$ such that $ab \notin E(G)$. Clearly, $(G + ab) - V(M)$ contains $S$ as a subset of $X'$ with $s = |S| = |N_{G'}(S)| + 1 = |N_{(G+ab)-V(M)}(S)| + 1$. Thus, $(G + ab) - V(M)$ has no perfect matching. Hence, $G + ab$ is not $k$-extendable. This contradicts the fact that $G$ is maximal non-$k$-extendable bipartite and proves that each vertex of $S$ is adjacent to every vertex of $(V(M) \cap Y) \cup N_{G'}(S)$. By a similar argument, one can establish that each vertex of $X\backslash S$ is adjacent to every vertex of $Y$. Consequently, each vertex of $(V(M) \cap Y) \cup N_{G'}(S)$ is adjacent to every vertex of $X$ and each vertex of $T = Y\backslash(V(M) \cup N_{G'}(S)) = \overline{N}_G(S) \cap Y$ is adjacent to every vertex of $X\backslash S$. Note that

$$|V(M) \cap X| + |X'\backslash S| = k + (n - k - s) = n - s,$$

$$|V(M) \cap Y| + |N_{G'}(S)| = k + s - 1$$

and

$$|T| = |\overline{N}_G(S) \cap Y| = n - (k + s - 1) = n - k - s + 1.$$

Hence, $G \cong K_{n,n} - \{xy | x \in S, y \in T\}$. Clearly, if $k+s=1$ or $n-s=0$, then $G$ is disconnected, contradicting the connectedness of $G$. Hence, $k+s \geqslant 2$ and $n-s \geqslant 1$. This completes the proof of our theorem. $\square$

**Remark 4.1.** Note that the maximal non-$k$-extendable bipartite graph $G$ in Theorem 4.3 is isomorphic to the graph

$$\overline{K}_s \vee \overline{K}_{k+s-1} \vee \overline{K}_{n-s} \vee \overline{K}_{n-k-s+1}.$$

As a corollary we have:

**Corollary 4.4.** *Let $G$ be a maximal non-$k$-extendable bipartite graph on $2n$ vertices, $1 \leqslant k \leqslant n-1$. Then $G$ is $(k-1)$-extendable.*

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